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A NICE GROUP STRUCTURE ON THE ORBIT SPACE OF UNIMODULAR ROWS-II

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Abstract. We establish an Excision type theorem for niceness of group structure on the orbit space of unimodular rows of length n modulo elementary action. This permits us to establish niceness for relative versions of results for the cases when $n = d + 1$; d being the dimension of the base algebra. We then study and establish niceness for the case when $n = d$, and also establish a relative version, when the base ring is a smooth affine algebra over an algebraically closed field.

1. Introduction

In Algebraic Topology the Excision Theorem is a useful theorem about relative homology: *viz.* given topological spaces X and subspaces A and U such that U is also a subspace of A, the theorem says that under certain circumstances, we can cut out (excise) U from both spaces such that the relative homologies of the pairs (X, A) and $(X \setminus U, A \setminus U)$ are isomorphic. Succinctly, Excision preserves homology; but it is known that it does not preserve homotopy.

Excision assists in computation of singular homology groups, as sometimes after excising an appropriately chosen subspace we obtain something easier to compute. Or, in many cases, it allows the use of induction. Coupled with the long exact sequence in homology, one can derive another useful tool for the computation of homology groups, the MayerVietoris sequence. In the axiomatic approach to homology, the theorem is the sixth of the Eilenberg-Steenrod axioms (See [2]).

In Algebra the above features of Excision were first introduced and studied by Milnor [10] in his book on Algebraic K-theory. In this context, Milnor introduced the double of a ring $R \times_I R$ of a ring w.r.t. an ideal I.

Later R.G. Swan studied in [22] whether Excision helped in computing the lower K-groups K_1, K_2; and showed that it failed.

The problem of characterizing the rings for which Excision holds was very important from the very beginning of the development of algebraic K-theory because of its relations to the Karoubi conjecture (on the equality of algebraic and topological K-theory groups of stable C^*-algebras), homology of congruence subgroups and other questions. In 1992 Suslin and M. Wodzicki in [19] solved the problem for rational algebraic K-theory. (Also see [20]).

But prior to that, there are two instances in Classical Algebraic K-theory where Suslin uses Excision for the linear group–refer ([11], Lemma 4.3); and the orthogonal group–refer ([15], Corollary 2.13) (where they prove that the relative orthogonal group $\mathrm{EO}_{2r}(R, I)$ is a normal subgroup of the orthogonal group $\text{Ø}_{2r}(R, I)$), and also to ([15], Lemma 2.14) where it is shown that one can deduce the injective stability for the relative orthogonal quotients to $\mathrm{K}_1\mathrm{O}(R, I)$ if one knows it for the ring and the double of the ring $R \times_I R$ w.r.t. the ideal I.

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In [25] W. van der Kallen defined a group structure for the orbits of unimodular rows of length $d+1$, where d was the dimension of the base ring, and studied the Excision property for orbit spaces $\mathrm{MSE}_n(R, I)$ of unimodular rows of length $n \geq 3$ modulo elementary action. (See Theorem 2.6). Later in [26] he showed that these orbit spaces also have a group structure when the size is a bit beyond half the dimension (the so-called Borsuk estimate).

In §3 we deduce a Double Excision theorem, which is a consequence of his theorem; but simplifies its usage. Using this we deduce in Theorem 3.6 that if I is an ideal in a ring R, and the orbit spaces $\mathrm{MSE}_n(R)$, and $\mathrm{MSE}_n(R, I)$ have the usual group structures (see [25, 26]), then the group structure on $\mathrm{MSE}_n(R, I)$ is nice (i.e. is Mennicke-like) if it is nice for the Excision ring $\mathrm{MSE}_n(R \oplus I)$. (We call this relative niceness criterion). (It would be interesting to know the appropriate analogue of the Double Excision theorem in Algebraic Topology.)

In [7] group structure on the orbit space was shown to be nice in the following cases:

1. Let A be an affine algebra (of dimension $d \geq 2$) over a perfect field k, where char $k \neq 2$ and the cohomological dimension $\mathrm{c.d.}_2\ k \leq 1$. Then the group structure on the orbit space $\mathrm{MSE}_{d+1}(A)$ is nice.
2. Let R be a commutative noetherian local ring of dimension $d \geq 3$, in which $2R = R$. Then the group structure on $\mathrm{MSE}_{d+1}(R[X])$ is nice.

We deduce from Double Excision that a relative version of the above results also hold. The key new observation here is Lemma 4.3 which asserts that if R is a local ring then the Excision ring $R \oplus I$ is also a local ring.

We then begin the study of niceness for the orbit spaces $\mathrm{Um}_d(A, I)$, when A is an affine algebra of dimension d over an algebraically closed field k. The key new input which allows us to study this case is the beautiful theorem of J. Fasel in ([4], Lemma 3.3) that for a smooth affine surface over an algebraically closed field of characteristic $\neq 2, 3$, a stably elementary 2×2 matrix is stably elementary symplectic. Another useful observation used is in [6] which asserts that if A is an affine threefold over an algebraically closed field then $\mathrm{Um}_4(A, (a)) = e_1 \mathrm{Sp}_4(A, (a))$, for $a \in A$.

If A is smooth, then we prove that the group structure on $\mathrm{Um}_d(A)/\mathrm{E}_d(A)$ is nice, when k is algebraically closed, and of characteristic different from $2, 3$.

Since $A \oplus I$ need not be smooth, even if A is smooth, we are unable to apply the relative niceness criterion here. However, we are able to circumvent this, and deduce the relative version, under the above assumptions on the algebra A, and the assumption that I is a principal ideal.

One of the interesting by-products of this paper is to get a relative Mennicke-Newman Lemma (see ([27], Lemma 3.2) for the absolute case, and ([25], Lemma 3.4) for the relative case when dealing with rows of length $d+1$, where d is the dimension of the base ring for the known cases earlier due to W.van der Kallen). It is the use of this version of the Mennicke–Newman lemma which permits us to study the concept of niceness for rows of smaller length and also to realize that the concept of niceness does not depend on 'which coordinate' in the relative case.

In all the cases we have shown the niceness of $\mathrm{MSE}_n(A, I)$ it is known that the stably free projective A-modules of rank $n-1$ are free − see [18], [5], [6] [12]; in fact, some essential ingredient in proving the freeness has been used by us to prove the niceness. In ([4], Theorem 2.1), J. Fasel has shown that when A is a smooth affine algebra of dimension $d \geq 3$ over a perfect field k with $\mathrm{c.d.}_2(k) \leq 2$ then $\mathrm{WMS}_{d+1}(A) = \mathrm{MS}_{d+1}(A)$; but by van der Kallen's theorem in [26] $\mathrm{MSE}_{d+1}(A) = \mathrm{WMS}_{d+1}(A)$; whence $\mathrm{MSE}_{d+1}(A)$ is nice. The result catches our attention as it is not known whether stably free projective A-modules of rank d are free for such affine algebras A when $\mathrm{c.d.}_2 k = 2$. We shall say more about this example in a sequel article.

2. Preliminaries

Throughout this note, R stands for a commutative ring with unity, for $n \geq 1$, $\mathrm{M}_n(R)$ the set of all $n \times n$ matrices over R and $\mathrm{GL}_n(R)$ the group of invertible $n \times n$ matrices over R. A row $v = (a_1, a_2, \ldots, a_n) \in R^n$ is said to be unimodular of length n, if there is a row $w = (b_1, b_2, \ldots, b_n) \in R^n$ such that $\langle v, w\rangle := v \cdot w^t = 1$, where w^t stands for the transpose of w. The set of all unimodular rows of length n over R will be denoted by $\mathrm{Um}_n(R)$. Given an ideal I of a ring R, let $\mathrm{Um}_n(R, I)$ denote the subset of $\mathrm{Um}_n(R)$ consisting of unimodular rows $v = (a_1, a_2, \ldots, a_n)$ with $v \equiv (1, 0, \ldots, 0)(\mathrm{mod}\ I)$ i.e., v is unimodular and $(a_1 - 1), a_2, \ldots, a_n \in I$. It can be shown that for any $v \in \mathrm{Um}_n(R, I)$ there exists $w \in \mathrm{Um}_n(R, I)$ such that $v \cdot w^t = 1$. Given $\lambda \in R$, for $i \neq j$, let $E_{ij}(\lambda) = I_n + \lambda e_{ij}$, where I_n denotes the identity matrix and $e_{ij} \in \mathrm{M}_n(R)$ is the matrix whose only non-zero entry is 1 at the (i, j)-th position. Such $E_{ij}(\lambda)$'s are called elementary matrices. The subgroup of $\mathrm{GL}_n(R)$ generated by $E_{ij}(\lambda), i \neq j, \lambda \in R$ is called the elementary subgroup of $\mathrm{GL}_n(R)$ and will be denoted by $\mathrm{E}_n(R)$. Similarly we define $\mathrm{E}_n(I)$ for any ideal I in R. We now recall the definition of the relative elementary group:

Definition 2.1. *Let I be an ideal of R. Then $\mathrm{E}_n(R, I)$ is defined to be the smallest normal subgroup of $\mathrm{E}_n(R)$ containing the element $E_{21}(x), x \in I$.*

For $n \geq 3$, the relative elementary group $\mathrm{E}_n(R, I)$ acts on the set of relative unimodular rows $\mathrm{Um}_n(R, I)$ and the orbit space of relative unimodular rows under relative elementary action is denoted by $\mathrm{Um}_n(R, I)/\mathrm{E}_n(R, I)$. We shall also use $\mathrm{MSE}_n(R, I)$ to denote the orbit space $\mathrm{Um}_n(R, I)/\mathrm{E}_n(R, I)$, following [26]. (When $I = R$, this is the orbit space $\mathrm{Um}_n(R)/\mathrm{E}_n(R)$.) Following is due to H.Bass.

Definition 2.2. (*Stable range condition $Sr_n(I)$*) *Let I be an ideal in R. We shall say stable range condition $Sr_n(I)$ holds for I if for any $(a_1, a_2, \ldots, a_{n+1})$ in $\mathrm{Um}_{n+1}(R, I)$ there exists c_i in I such that $(a_1 + c_1 a_{n+1}, a_2 + c_2 a_{n+1}, \ldots, a_n + c_n a_{n+1}) \in \mathrm{Um}_n(R, I)$.*

We recall the following argument of Vaserstein (see [24]) for an ideal I in R. Assume $Sr_n(I)$ holds for I. Let $(a_1, a_2, \ldots, a_{n+2}) \in \mathrm{Um}_{n+2}(R, I)$. Then there exists $(b_1, b_2, \ldots, b_{n+2}) \in Um_{n+2}(R, I)$ such that $\Sigma_{i=1}^{i=n+2} a_i b_i = 1$. So $(a_1, a_2, \ldots, a_n, a_{n+1}b_{n+1} + a_{n+2}b_{n+2}) \in \mathrm{Um}_n(R, I)$. Now by the condition $Sr_n(I)$ on I we have c_i's $\in I$ such that $(a_i + c_i\{a_{n+1}b_{n+1} + a_{n+2}b_{n+2}\}) \in \mathrm{Um}_n(R, I)$. In particular $(a_1 + c_1\{a_{n+1}b_{n+1} + a_{n+2}b_{n+2}\}, a_2 + c_2\{a_{n+1}b_{n+1} + a_{n+2}b_{n+2}\}, \ldots, a_n + c_n\{a_{n+1}b_{n+1} + a_{n+2}b_{n+2}\}, a_{n+1}) \in \mathrm{Um}_{n+1}(R, I)$. Subtracting suitable multiples of a_{n+1} from first n coordinates we have $(a_1 + c_1 a_{n+2}b_{n+2}, a_2 + c_2 a_{n+2}b_{n+2}, \ldots, a_n + c_n a_{n+2}b_{n+2}, a_{n+1}) \in \mathrm{Um}_{n+1}(R, I)$. Therefore the condition $Sr_n(I)$ implies the condition $Sr_{n+1}(I)$.

Definition 2.3. (Stable range $Sr(I)$, Stable dimension $Sd(I)$) *We shall define the stable range of I denoted by $Sr(I)$ to be the least integer n such that $Sr_n(I)$ holds for I. We shall define stable dimension of I by $Sd(I) = Sr(I) - 1$.*

Following is proved in [24].

Lemma 2.4. (Vaserstein) *Let I, J be two ideals in R such that $I \subset J$. Then the following are true.*
(a) $Sr(I) \leq Sr(J)$.
(b) $Sr(J/I) \leq Sr(J)$.
In particular we have $Sr(I) \leq Sr(R)$ and $Sr(R/I) \leq Sr(R)$. The above assertions are also true for stable dimension.

Definition 2.5. (Excision Ring) *Let R be a ring and I an ideal in R. The Excision ring $\mathbb{Z} \oplus I$, has coordinate-wise addition and multiplication given by: $(m, i) \cdot (n, j) = (mn, mj + ni + ij)$. The additive identity of this ring is $(0, 0)$ and the multiplicative identity is $(1, 0)$. We have a ring homomorphism*

$f : \mathbb{Z}\oplus I \to R$ defined by $f(n,i) = n+i$ which will induce a map $\mathfrak{f} : \mathrm{Um}_n(\mathbb{Z}\oplus I, 0\oplus I) \longrightarrow \mathrm{Um}(R\oplus I, 0\oplus I)$ defined by $(a_i) \mapsto (f(a_i))$.

We recall Excision theorem (see [25], Theorem 3.21):

Theorem 2.6. (Excision theorem) *Let $n \geq 3$ be an integer and I be an ideal in a commutative ring R. Then the natural maps $F : \mathrm{MSE}_n(\mathbb{Z}\oplus I, 0\oplus I) \to \mathrm{MSE}_n(R, I)$ defined by $[(a_i)] \mapsto [(f(a_i))]$ and $G : \mathrm{MSE}_n(\mathbb{Z}\oplus I, 0\oplus I) \to \mathrm{MSE}_n(\mathbb{Z}\oplus I)$ defined by $[(a_i)] \mapsto [(a_i)]$ are bijections.*

Definition 2.7. (*Vaserstein, van der Kallen's rule or Vv rule*) *Let $v, w \in Um_n(R)$, $n \geq 3$ be given by $v = (a, a_2, a_3, \ldots, a_n)$, $w = (b, a_2, a_3, \ldots, a_n)$. We shall say that a group operation $*$ on $\mathrm{MSE}_n(R)$ is given by Vaserstein, van der Kallen's rule if it satisfies one of the following equivalent conditions (see remark following Theorem 3.6 in [25]).*

1 *Choose $p \in R$ such that $ap \equiv 1 \,(\mathrm{mod}\ (a_2, a_3, \ldots a_n))$. Then*
$$[w] * [v] = [(a(b+p) - 1, a_2(b+p), a_3, \ldots, a_n)].$$
2. *Let $\alpha \in \mathrm{M}_2(R)$ such that $e_1\alpha = (a, a_2)$ and $\overline{\alpha} \in \mathrm{GL}_2(\overline{R})$; $\overline{R} = R/(a_3, a_4, \ldots, a_n)$. Then*
$$[w] * [v] = [((b, a_2)\alpha, a_3, \ldots, a_n)].$$

A group operation $*$ on $\mathrm{MSE}_n(R, I)$ is said to be given by Vaserstein, van der Kallen's rule if the induced operation on $\mathrm{MSE}_n(\mathbb{Z}\oplus I)$ by F, G (see Theorem 2.6) follows Vaserstein, van der Kallen's rule in the previous sense. We shall abbreviate it as Vv rule.

Definition 2.8. (Universal weak Mennicke symbol $\mathrm{WMS}_n(R), n \geq 2$) (*cf.* [26], Section 3) *We define the universal weak Mennicke symbol on $\mathrm{MSE}_n(R)$ by a set map $\mathrm{wms} : \mathrm{MSE}_n(R) \longrightarrow \mathrm{WMS}_n(R)$, $[v] \longmapsto \mathrm{wms}(v)$ to a group $\mathrm{WMS}_n(R)$. The group $\mathrm{WMS}_n(R)$ is the free group generated by $\mathrm{wms}(v), v \in \mathrm{Um}_n(R)$ modulo the following relations*

(1) $\mathrm{wms}(v) = \mathrm{wms}(vg)$ *if $g \in \mathrm{E}_n(R)$.*
(2) If $(q, v_2, \ldots, v_n), (1+q, v_2, \ldots, v_n) \in \mathrm{Um}_n(R)$ and $r(1+q) = q(\mathrm{mod}\ (v_2, \ldots, v_n))$, then $\mathrm{wms}(q, v_2, \ldots, v_n) = \mathrm{wms}(r, v_2, \ldots, v_n)\mathrm{wms}(1+q, v_2, \ldots, v_n)$.

In ([26], Lemma 3.5) van der Kallen has shown that if $[v], [w] \in \mathrm{MSE}_n(R), n \geq 3, v = (a, a_2, a_3 \ldots, a_n)$, $w = (b, a_2, a_3 \ldots, a_n)$ and $p \in R$ such that $ap \equiv 1$ (mod $(a_2, a_3, \ldots a_n)$), then
$$\mathrm{wms}(w)\mathrm{wms}(v) = \mathrm{wms}((a(b+p) - 1, a_2(b+p), a_3, \ldots, a_n)).$$
We recall ([26], Theorem 4.1).

Theorem 2.9. (W. van der Kallen) *Let R be a ring of stable dimension d, $d \leq 2n-4$ and $n \geq 3$. Then the universal weak Mennicke symbol $\mathrm{wms} : \mathrm{MSE}_n(R) \longrightarrow \mathrm{WMS}_n(R)$ is bijective and $\mathrm{WMS}_n(R)$ has the structure of an abelian group given by Vv rule.*

Remark 2.10. *Let I be an ideal in a ring R, with $max(R)$ a disjoint union of $V(I)$ and finitely many subsets V_i each a noetherian topological space of dimension at most d. Then the maximal spectrum of $(\mathbb{Z}\oplus I)$ is the union of finitely many subspaces of dimension atmost d whenever $d \geq 2$. Therefore $\mathbb{Z}\oplus I$ has stable dimension at most d for $d \geq 2$* (see [25], 3.19). *So by the above Theorem $\mathrm{MSE}_n(\mathbb{Z}\oplus I)$ has a group structure given by Vv rule whenever $n \geq max\{3, 2+\frac{d}{2}\}$. Equivalently $\mathrm{MSE}_n(R, I)$ has a group structure given by Vv rule whenever $n \geq 2+\frac{d}{2}, d \geq 2$. (The group structure on $MSE_n(R, I)$ is inherited from the one on the subgroup $MSE_n(\mathbb{Z}\oplus I, 0\oplus I)$ of $MSE_n(\mathbb{Z}\oplus I)$).*

Definition 2.11. *We say that the group structure $\mathrm{MSE}_n(R)$, $n \geq 3$, given by Vv rule is* **nice**, *if it is given by the 'coordinate-wise multiplication' formula:*
$$[(b, a_2, \ldots, a_n)] * [(a, a_2, \ldots, a_n)] = [(ab, a_2, \ldots, a_n)].$$

A group structure on $\mathrm{MSE}_n(R, I)$ *given by* Vv *rule is said to be* **nice** *if it satisfies one of the following equivalent conditions .*

(1) $[(b, a_2, \ldots, a_n)] * [(a, a_2, \ldots, a_n)] = [(ab, a_2, \ldots, a_n)]$.
(2) $[(a_1, \ldots, a_{n-1}, b)] * [(a_1, \ldots, a_{n-1}, a)] = [(a_1, \ldots, a_{n-1}, ab)]$.
(3) $[(a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n)]*[(a_1, \ldots, a_{i-1}, a, a_{i+1}, \ldots, a_n)] = [(a_1, \ldots, a_{i-1}, ab, a_{i+1}, \ldots, a_n)]$ *for* $i = 2, 3, \ldots, n$.

Proof of the equivalence:
Given $i \in I$ we use $\tilde{i}$ to denote $(0, i) \in \mathbb{Z} \oplus I$. If $x = n + i, n \in \mathbb{Z}, i \in I$ then we define $\tilde{x} = (n, i) \in \mathbb{Z} \oplus I$ to be a preimage of x under f (see Definition 2.5). Equivalence of (2) and (3) is obvious. So assume that (3) holds. Let $(p, b_2, b_3, \ldots, b_n) \in \mathrm{Um}_n(R, I)$ be such that $ap + \Sigma_{i=2}^n a_i b_i = 1$. Now in $\mathrm{MSE}_n(R, I)$ we have

$$\begin{aligned}
&[(b, a_2, \ldots, a_n)] * [(a, a_2, \ldots, a_n)] \\
&= [(a(b+p) - 1, a_2(b+p), a_3, \ldots, a_n)] \\
&= [(a(b+p) - 1, a_2\lambda(b+p), a_3, \ldots, a_n)];\ \text{ assume } a(b+p) - 1 = 1 - \lambda, \lambda \in I \\
&= [(a(b+p) - 1, a_2, a_3, \ldots, a_n)] * [(a(b+p) - 1, \lambda(b+p), a_3, \ldots, a_n)] \\
&= [(a(b+p) - 1, a_2, a_3, \ldots, a_n)] * [e_1] \\
&= [(ab, a_2, \ldots, a_n)].
\end{aligned}$$

Note that $(\tilde{a}(\tilde{b} + \tilde{p}) - 1, \tilde{\lambda}(\tilde{b} + \tilde{p}), \tilde{a_3}, \ldots, \tilde{a_n})$ in $\mathrm{Um}_n(\mathbb{Z} \oplus I, 0 \oplus I)$ can be completed to a matrix in $\mathrm{E}_n(\mathbb{Z} \oplus I)$ and therefore to a matrix in $\mathrm{E}_n(\mathbb{Z} \oplus I, 0 \oplus I)$ by Excision theorem 2.6. So its image $(a(b+p)-1, \lambda(b+p), a_3, \ldots, a_n) \in \mathrm{Um}_n(R, I)$ under f can be completed to a matrix in $\mathrm{E}_n(R, I)$. This proves (1).

Now we assume that (1) holds. Then $[(\tilde{a_1}, \ldots, \tilde{a_{n-1}}, \tilde{b})], [(\tilde{a_1}, \ldots, \tilde{a_{n-1}}, \tilde{a})] \in \mathrm{MSE}_n(\mathbb{Z} \oplus I, 0 \oplus I)$ correspond to $[(a_1, \ldots, a_{n-1}, b)], [(a_1, \ldots, a_{n-1}, a)]$ by F (see Theorem 2.6) respectively. We choose $(\tilde{b_1}, \tilde{b_2}, \ldots, \tilde{b_{n-1}}, \tilde{p}) \in \mathrm{Um}_n(\mathbb{Z} \oplus I, 0 \oplus I)$ such that $\Sigma_{i=1}^{n-1} \tilde{a_i}\tilde{b_i} + \tilde{a}\tilde{p} = 1$. Since the induced operation on $\mathrm{MSE}_n(\mathbb{Z} \oplus I)$ satisfies Vv rule we have

$$\begin{aligned}
&[(\tilde{a_1}, \ldots, \tilde{a_{n-1}}, \tilde{b})] * [(\tilde{a_1}, \ldots, \tilde{a_{n-1}}, \tilde{a})] \\
&= [(\tilde{a_1}(\tilde{b} + \tilde{p}), \tilde{a_2}, \ldots, \tilde{a_{n-1}}, \tilde{a}(\tilde{b} + \tilde{p}) - 1)] \\
&= [(\tilde{a_1}(\tilde{b} + \tilde{p} + 1 - \tilde{a}(\tilde{b} + \tilde{p})), \tilde{a_2}, \ldots, \tilde{a_{n-1}}, \tilde{a}(\tilde{b} + \tilde{p}) - 1)] \\
&= [(\tilde{a_1}(\tilde{b} + \tilde{p} + 1 - \tilde{a}(\tilde{b} + \tilde{p})), \tilde{a_2}, \ldots, \tilde{a_{n-1}}, \tilde{a}(\tilde{b} + \tilde{p}) - 1 + \tilde{a_1}(\tilde{b} + \tilde{p} + 1 - \tilde{a}(\tilde{b} + \tilde{p})))] \ldots\ldots (a).
\end{aligned}$$

Now both $(\tilde{a_1}, \tilde{a_2}, \ldots, \tilde{a_{n-1}}, \tilde{a}(\tilde{b}+\tilde{p})-1+\tilde{a_1}(\tilde{b}+\tilde{p}+1-\tilde{a}(\tilde{b}+\tilde{p})))$ and $(\tilde{b}+\tilde{p}+1-\tilde{a}(\tilde{b}+\tilde{p}), \tilde{a_2}, \ldots, \tilde{a_{n-1}}, \tilde{a}(\tilde{b}+\tilde{p}) - 1 + \tilde{a_1}(\tilde{b}+\tilde{p}+1-\tilde{a}(\tilde{b}+\tilde{p})))$ are in $\mathrm{MSE}_n(\mathbb{Z} \oplus I, 0 \oplus I)$. Then by our given hypothesis (1) equation (a) equals

$$\begin{aligned}
&[(\tilde{a_1}, \tilde{a_2}, \ldots, \tilde{a_{n-1}}, \tilde{a}(\tilde{b} + \tilde{p}) - 1 + \tilde{a_1}(\tilde{b} + \tilde{p} + 1 - \tilde{a}(\tilde{b} + \tilde{p})))] * [(\tilde{b} + \tilde{p} + 1 - \tilde{a}(\tilde{b} + \tilde{p}), \tilde{a_2}, \ldots, \tilde{a_{n-1}}, \\
&\tilde{a}(\tilde{b} + \tilde{p}) - 1 + \tilde{a_1}(\tilde{b} + \tilde{p} + 1 - \tilde{a}(\tilde{b} + \tilde{p})))] \\
&= [(\tilde{a_1}, \tilde{a_2}, \ldots, \tilde{a_{n-1}}, \tilde{a}(\tilde{b} + \tilde{p}) - 1)] * [(\tilde{b} + \tilde{p} + 1 - \tilde{a}(\tilde{b} + \tilde{p}), \tilde{a_2}, \ldots, \tilde{a_{n-1}}, \tilde{a}(\tilde{b} + \tilde{p}) - 1)] \\
&= [(\tilde{a_1}, \tilde{a_2}, \ldots, \tilde{a_{n-1}}, \tilde{a}(\tilde{b} + \tilde{p}) - 1)] * [(\tilde{b} + \tilde{p}, \tilde{a_2}, \ldots, \tilde{a_{n-1}}, \tilde{a}(\tilde{b} + \tilde{p}) - 1)] \\
&= [(\tilde{a_1}, \tilde{a_2}, \ldots, \tilde{a_{n-1}}, \tilde{a}(\tilde{b} + \tilde{p}) - 1)] * [e_1] \\
&= [(\tilde{a_1}, \tilde{a_2}, \ldots, \tilde{a_{n-1}}, \tilde{a}\tilde{b})];\ \text{since } \tilde{a}\tilde{p} \equiv 1 \pmod{(\tilde{a_1}, \tilde{a_2}, \ldots, \tilde{a_{n-1}})} \ldots\ldots (b).
\end{aligned}$$

Comparing both sides of the equation $[(\tilde{a_1}, \ldots, \tilde{a_{n-1}}, \tilde{b})] * [(\tilde{a_1}, \ldots, \tilde{a_{n-1}}, \tilde{a})] = [(\tilde{a_1}, \tilde{a_2}, \ldots, \tilde{a_{n-1}}, \tilde{a}\tilde{b})]$ under F in $\mathrm{MSE}_n(R, I)$ we get (2).

3. On van der Kallen's Excision theorem

In this section we recall the construction and properties of the Excision ring. Let R be a ring and I an ideal in R. The Excision ring $R \oplus I$, has coordinate-wise addition and multiplication given by: $(r,i)\cdot(s,j) = (rs, rj+si+ij)$. The additive identity of this ring is $(0,0)$ and the multiplicative identity is $(1,0)$. We use the Excision Theorem to prove:

Lemma 3.1. (*Double Excision*) *Let R be a ring and I an ideal in R. Under the natural maps, for $n \geq 3$, the following orbit spaces are in bijection:*

$$\mathrm{MSE}_n(R \oplus I, 0 \oplus I) \leftrightarrow \mathrm{MSE}_n(\mathbb{Z} \oplus (0 \oplus I), (0 \oplus I)) \leftrightarrow \mathrm{MSE}_n(\mathbb{Z} \oplus (0 \oplus I))$$

$$\leftrightarrow \mathrm{MSE}_n(\mathbb{Z} \oplus I) \leftrightarrow \mathrm{MSE}_n(\mathbb{Z} \oplus I, 0 \oplus I) \leftrightarrow \mathrm{MSE}_n(R, I).$$

Let $\pi_2 : R \oplus I \to R$ be the surjective map given by $\pi_2(a,i) = a+i$. Assume that for $n \geq 3$, there exist group structures (with product given by the van der Kallen formula) on the orbit spaces $\mathrm{MSE}_n(R,I)$ and $\mathrm{MSE}_n(R \oplus I, 0 \oplus I)$ then, $\pi_2 : \mathrm{MSE}_n(R \oplus I, 0 \oplus I) \to \mathrm{MSE}_n(R,I)$ is a group homomorphism. In particular, π_2 preserves the nice group structure.

Proof. That there is a bijection between the first three orbit spaces listed above follows from Excision theorem 2.6. Similarly, that there is a bijection between the last three orbit spaces also follows from Excision theorem.

It only remains to check that there is bijection between $\mathrm{MSE}_n(\mathbb{Z} \oplus 0 \oplus I)$ and $\mathrm{MSE}_n(\mathbb{Z} \oplus I)$. But this follows from the fact that $\varphi : \mathbb{Z} \oplus (0 \oplus I) \to \mathbb{Z} \oplus I$ by $\varphi((m,0,i)) = (m,i)$ is a ring isomorphism inducing an isomorphism in MSE_n.

The last assertion regarding group homomorphism follows from the fact that π_2 respects the ring structure of the Excision ring. $\square$

Definition 3.2. *We shall say a ring homomorphism $\phi : B \longrightarrow D$ is a retract if there exists a ring homomorphism $\gamma : D \longrightarrow B$ so that $\phi \circ \gamma$ is identity on D. We shall also say that D is a retract of B.*

Note that if $\phi : B \longrightarrow D$ is a retract then ϕ induces an onto map from $\mathrm{Um}_n(B)$ to $\mathrm{Um}_n(D)$. We recall a Lemma of Suslin (see [11], Lemma 4.3), which gives a handle on the relative elementary group in certain special cases.

Lemma 3.3. *Let B, D be rings and let D be a retract of B and let $\pi : B \twoheadrightarrow D$. If $J = \ker(\pi)$, then* $\mathrm{E}_n(B,J) = \mathrm{E}_n(B) \cap \mathrm{SL}_n(B,J), n \geq 3$.

We isolate here another result which is a consequence of Lemma 3.3 above and which will be used repeatedly throughout this paper.

Lemma 3.4. *Let the quotient map $q : R \longrightarrow R/I$ be a retract. Let $v \in \mathrm{Um}_n(R,I)$ be such that its class $[v]$ is trivial in $\mathrm{MSE}_n(R), n \geq 3$. Then $[v]$ is also trivial in $\mathrm{MSE}_n(R,I)$.*

Proof. We have a ring homomorphism $f : R/I \longrightarrow R$ such that $q \circ f = id$. By hypothesis there exists a $\varepsilon \in \mathrm{E}_n(R)$ such that $v\varepsilon = e_1$. Taking the image in R/I we have $e_1 q(\varepsilon) = e_1$ and therefore $e_1 f \circ q(\varepsilon) = e_1$. Let $\varepsilon' = \varepsilon(f \circ q(\varepsilon))^{-1} \in \mathrm{E}_n(R) \cap \mathrm{SL}_n(R,I)$. Then by Lemma 3.3 we have $\varepsilon' \in \mathrm{E}_n(R,I)$ and $v\varepsilon' = e_1$ holds obviously. So $[v]$ is trivial in $\mathrm{MSE}_n(R,I)$. $\square$

A special case of the above lemma says the following.

Corollary 3.5. *Let R be a ring and I be an ideal in R and $n \geq 3$ be an integer. If $[v] \in \mathrm{Um}_n(R \oplus I, 0 \oplus I)$ is such that $[v] = [e_1]$ in $\mathrm{MSE}_n(R \oplus I)$, then $[v] = [e_1]$ in $\mathrm{MSE}_n(R \oplus I, 0 \oplus I)$.*

Lemma 3.6. (*Relative Niceness Criterion*)

Let $R \oplus I$ be the Excision ring of R with respect to an ideal I in R and $n \geq 3$. Suppose both $\mathrm{MSE}_n(R, I)$ *and* $\mathrm{MSE}_n(R \oplus I)$ *have group structures given by Vv rule. Then the group structure on* $\mathrm{MSE}_n(R, I)$ *is nice whenever it is nice for* $\mathrm{MSE}_n(R \oplus I)$*.*

Proof. Corollary 3.5 shows that the map $\phi : \mathrm{MSE}_n(R \oplus I, 0 \oplus I) \longrightarrow \mathrm{MSE}_n(R \oplus I)$ sending the relative class of a unimodular row $v \in \mathrm{Um}_n(R \oplus I, 0 \oplus I)$ to its absolute class is an injective group homomorphism. So if the group structure on $\mathrm{MSE}_n(R \oplus I)$ is nice then it is so on $\mathrm{MSE}_n(R \oplus I, 0 \oplus I)$ also. Now by Double Excision Lemma 3.1 we have $\mathrm{MSE}_n(R \oplus I, 0 \oplus I) = \mathrm{MSE}_n(R, I)$. So the result follows. $\square$

By Theorem 3.1 we have $\mathrm{MSE}_n(R \oplus I, 0 \oplus I) \cong \mathrm{MSE}_n(\mathbb{Z} \oplus I)$. So Corollary 3.5 leads us to ask the following.

Question 3.7. *Is the map* $\mathrm{MSE}_n(\mathbb{Z} \oplus I) \longrightarrow \mathrm{MSE}_n(R \oplus I), n \geq 3$ *injective?*

Lemma 3.6 leads us to ask the following.

Question 3.8. *Is it true that the group structure on* $\mathrm{MSE}_n(\mathbb{Z} \oplus I, 0 \oplus I)$ *is nice if and only if the group structure on* $\mathrm{MSE}_n(\mathbb{Z} \oplus I)$ *is nice, when both have a group structure given by Vv rule?*

4. Relative orbit space: size $(d+1)$

It is well known that the double of a ring w.r.t. an ideal is the same as the Excision ring w.r.t. an ideal I. The reader may look at the reference below for details, if necessary.

Proposition 4.1. (*cf.* [9], Proposition 3.1) *Let R be a ring of dimension d and I a finitely generated ideal of R.*

Consider the Cartesian square:

$$\begin{array}{ccc} C & \longrightarrow & R \\ \big\downarrow & & \big\downarrow \\ R & \longrightarrow & R/I \end{array}$$

Then, C is finitely generated algebra of dimension d over R and integral over R. In fact, $C \simeq R \oplus I$ with coordinate wise addition and multiplication defined by $(a, i)(b, j) = (ab, aj + ib + ij)$.

In particular, if R is an affine algebra of dimension d over a field k, then C is also an affine algebra of dimension d over k.

Theorem 4.2. *Let A be an affine algebra of dimension $d \geq 2$ over a perfect field k, with* $\mathrm{char}\ k \neq 2$ *and the cohomological dimension* $\mathrm{c.d.}_2\ k \leq 1$*. Let I be an ideal of A. Then the group structure on* $\mathrm{MSE}_{d+1}(A, I)$ *is nice.*

Proof. By Lemma 3.6 it is enough to prove that the group structure on $\mathrm{MSE}_{d+1}(A \oplus I)$ is nice. Now $A \oplus I$ is an affine algebra of dimension d over k by Proposition 4.1. So the result follows from ([7], Theorem 3.9). $\square$

Lemma 4.3. *Let $(R, \mathfrak{m})$ be a local ring with maximal ideal $\mathfrak{m}$. Then the Excision ring $R \oplus I$ with respect to a proper ideal I in R is also a local ring with maximal ideal $\mathfrak{m} \oplus I$.*

Proof. $R \oplus I$ is a commutative ring with identity $(1, 0)$. For any $i \in I \subset \mathfrak{m}$, $1 + i$ is a unit in R with inverse of the form $1 + j$ for some $j \in I$. Therefore $(1, 0) + (0, i) = (1, i)$ is a unit in $R \oplus I$ with inverse $(1, j)$. So $0 \oplus I$ is contained in the Jacobson radical of $R \oplus I$. We also have $\mathfrak{m} \oplus 0$ contained in the Jacobson radical since any element in $(1, 0) + \mathfrak{m} \oplus 0$ is a unit in $R \oplus I$. So $\mathfrak{m} \oplus I$ is contained in the Jacobson radical. But $\mathfrak{m} \oplus I$ is a maximal ideal in $R \oplus I$. Hence the result follows. $\square$

Theorem 4.4. *Let $(R, \mathfrak{m})$ be a commutative, noetherian, local ring of dimension $d \geq 2$, in which $2R = R$. Let I be a proper ideal in R. Then the group structure on $\mathrm{MSE}_{d+1}(R[X], I[X])$ is nice.*

Proof. By Lemma 3.6 it is enough to prove that the group structure on $\mathrm{MSE}_{d+1}((R \oplus I)[X])$ is nice. But $R \oplus I$ is a local ring by Lemma 4.3. So the result follows from Theorem 5.1 in [7]. $\square$

We recall ([26], Theorem 2.2) of van der Kallen.

Theorem 4.5. (*W. van der Kallen*)
Let $n \geq 3$. Assume that R is commutative with $Sd(R) \leq 2n-3$ or assume that the maximal spectrum of R is the union of finitely many noetherian subspaces of dimension at most $2n-3$. Let i, j be non-negative integers. For every $\sigma \in \mathrm{GL}_{n+i}(R) \cap \mathrm{E}_{n+i+j+1}(R, I)$ there are matrices u, v, w, M with entries in I and q with entries in R such that

$$\begin{pmatrix} I_{i+1} + uq & v \\ wq & I_{n-1} + M \end{pmatrix} \in \sigma \mathrm{E}_{n+i}(R, I), \quad \begin{pmatrix} I_{j+1} + qu & qv \\ w & I_{n-1} + M \end{pmatrix} \in \mathrm{E}_{n+j}(R, I).$$

Corollary 4.6. *Let A be an affine algebra of dimension $d \geq 2$ over an algebraically closed perfect field k, with* char $k \neq 2$ *and the cohomological dimension* c.d.$_2$ $k \leq 1$. *Let $\sigma \in \mathrm{SL}_{d+1}(A, I) \cap \mathrm{E}_{d+2}(A, I)$. Then, $[e_1\sigma] = [e_1]$ in $\mathrm{MSE}_{d+1}(A, I)$ i.e. $e_1\sigma$ is relatively elementarily equivalent to e_1.*

Proof. Putting $i = j = 0$ and $n = d + 1$ in the Theorem 4.5 we have

$$\begin{pmatrix} 1 + uq & v \\ wq & I_d + M \end{pmatrix} \in \sigma \mathrm{E}_{d+1}(R, I), \quad \begin{pmatrix} 1 + qu & qv \\ w & I_d + M \end{pmatrix} \in \mathrm{E}_{d+1}(R, I).$$

Therefore in $\mathrm{MSE}_{d+1}(R, I)$ we have

$$\begin{aligned} &[e_1\sigma] \\ &\quad = [(1 + uq, v)] \\ &\quad = [(1 + uq, v)] * [(1 + uq, q)]; \quad \text{since } [(1 + uq, q)] \text{ is the identity } [e_1] \\ &\quad = [(1 + uq, qv)]; \quad \text{since the group structure on } \mathrm{MSE}_{d+1}(R, I) \text{ is nice by Theorem 4.2} \\ &\quad = [e_1]. \end{aligned}$$

$\square$

Similarly using Theorem 4.4 we have the following.

Corollary 4.7. *Let $(R, \mathfrak{m})$ be a commutative, noetherian, local ring of dimension $d \geq 2$, in which $2R = R$ and $\sigma(X) \in \mathrm{SL}_{d+1}(R[X], I[X]) \cap \mathrm{E}_{d+2}(R[X], I[X])$. Then, $e_1\sigma(X)$ is relatively elementarily equivalent to e_1.*

5. Improved injective stability in relative case

In this section we shall recall the following relative version of ([13], Theorem 3.4) with respect to a principal ideal.

Lemma 5.1. (*c.f.* [8]) *Let A be an affine algebra of dimension $d \geq 2$ over an algebraically closed field k and $I = (a)$ a principal ideal in A. Let $\alpha \in \mathrm{SL}_{d+1}(A, I) \cap \mathrm{E}(A, I)$. Then α is isotopic to identity relative to I. Moreover if A is nonsingular then,*

$$\mathrm{SL}_{d+1}(A, I) \cap \mathrm{E}(A, I) = \mathrm{E}_{d+1}(A, I).$$

Lemma 5.2. *Let R be a commutative ring and I an ideal in R. Let $u, v \in \mathrm{Um}_3(R, I)$ such that $u\alpha = v$ for some $\alpha \in \mathrm{SL}_3(R, I) \cap \mathrm{E}_4(R, I)$. Then u and v are elementary equivalent relative to I.*

Proof. Let $\mathrm{E}_n^k(R,I)$ be the subgroup of $\mathrm{GL}_n(R)$ generated by the $E_{ki}(a)$ with $a \in R, i \neq k$ and the $E_{ik}(x), x \in I, i \neq k$. In ([25], Lemma 2.2) it has been shown that $\mathrm{E}_n(R,I) = \mathrm{E}_n^1(R,I) \cap \mathrm{GL}_n(R,I)$. Let σ be the permutation matrix obtained by interchanging the first and kth row of I_n. Then

$$\mathrm{E}_n(R,I) = \sigma \mathrm{E}_n(R,I)\sigma^{-1} = \sigma \mathrm{E}_n^1(R,I)\sigma^{-1} \cap \mathrm{GL}_n(R,I) = \mathrm{E}_n^k(R,I) \cap \mathrm{GL}_n(R,I).$$

In particular any matrix in $\mathrm{E}_4(R,I)$ can be expressed as product of elementary matrices of the form $E_{4i}(a); a \in R, 1 \leq i \leq 3$ and $E_{i4}(x); x \in I, 1 \leq i \leq 3$.

We shall show that $u\alpha \in u\mathrm{E}_3(R,I)$. Let $u = (u_1, u_2, u_3)$ and $w = (w_1, w_2, w_3) \in \mathrm{Um}_3(R,I)$ be such that $u_1w_1 + u_2w_2 + u_3w_3 = 1$. Define

$$\theta(w,u) = \begin{pmatrix} 0 & -u_1 & -u_2 & -u_3 \\ u_1 & 0 & -w_3 & w_2 \\ u_2 & w_3 & 0 & -w_1 \\ u_3 & -w_2 & w_1 & 0 \end{pmatrix}.$$

We have $\left(\begin{smallmatrix} 1 & 0 \\ 0 & \alpha \end{smallmatrix}\right) \in \mathrm{E}_4(R,I)$ and $\left(\begin{smallmatrix} 1 & 0 \\ 0 & \alpha \end{smallmatrix}\right)^t \theta(w,u) \left(\begin{smallmatrix} 1 & 0 \\ 0 & \alpha \end{smallmatrix}\right) = \theta(w', u\alpha)$, for some $w' \in \mathrm{Um}_3(R,I)$. Now $\left(\begin{smallmatrix} 1 & 0 \\ 0 & \alpha \end{smallmatrix}\right)^t \in \mathrm{E}_4(R,I)$ and therefore is product of elementary matrices of the form $E_{4i}(a); a \in R, 1 \leq i \leq 3$ and $E_{i4}(x); x \in I, 1 \leq i \leq 3$. Let $\beta\theta(w,u)\beta^t = \theta(w', u\hat{\beta})$ for such elementary matrix β. We shall compute $\hat{\beta}$ when β is elementary matrices of different type as described above. So first assume $\beta = E_{14}(x), x \in I$. Then

$$(u_1 + xw_2, u_2 - xw_1, u_3) = u \begin{pmatrix} 1 + xw_1w_2 & -xw_1^2 & 0 \\ xw_2^2 & 1 - xw_1w_2 & 0 \\ xw_2w_3 & -xw_1w_3 & 1 \end{pmatrix} \in u \begin{pmatrix} I_2 + \nu\mu & 0 \\ I \times I & 1 \end{pmatrix}$$

for $\nu = (w_1, w_2)^t$ and $\mu = (xw_2, -xw_1)$. Note that $\mu\nu = 0$. So we choose $\hat{\beta} = \begin{pmatrix} I_2 + \nu\mu & 0 \\ I \times I & 1 \end{pmatrix} \in \mathrm{E}_3(A,I)$ by ([23], Lemma 1.1(b)). In the other cases finding $\hat{\beta}$ is easy. We have

$$\hat{\beta} = \begin{cases} I_3 & \text{when } \beta = E_{41}(a), a \in R \\ \gamma^t & \text{when } \beta = E_{4i}(a), i = 2,3, a \in R \text{ or } E_{i4}(x), i = 2,3, x \in I \ \textit{i.e.}\ \beta \text{ is of the form } \left(\begin{smallmatrix} 1 & 0 \\ 0 & \gamma \end{smallmatrix}\right). \end{cases}$$

So we have $\theta(w', u\alpha) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \alpha \end{smallmatrix}\right)^t \theta(w,u) \left(\begin{smallmatrix} 1 & 0 \\ 0 & \alpha \end{smallmatrix}\right) = \theta(w', u\hat{\alpha})$ where $\hat{\alpha} = \prod \hat{\beta}$. Clearly $\hat{\alpha}^t \in \mathrm{E}_3^3(R,I)$. It is easy to see that $\hat{\alpha}^t \in \mathrm{GL}_3(R,I)$ since $\left(\begin{smallmatrix} 1 & 0 \\ 0 & \alpha \end{smallmatrix}\right)^t = \prod \beta \in \mathrm{GL}_3(R,I)$. So $\hat{\alpha}^t \in \mathrm{E}_3(R,I)$ and therefore $\hat{\alpha} \in \mathrm{E}_3(R,I)$. Thus $v = u\alpha = u\hat{\alpha} \in u\mathrm{E}_3(R,I)$. □

Theorem 5.3. *Let R be any commutative ring of dimension 3 and I an ideal in R such that $\mathrm{SL}_4(R,I) \cap \mathrm{E}(R,I) = \mathrm{E}_4(R,I)$. Then $\mathrm{MSE}_3(R,I)$ has an abelian Witt group structure given by Vv rule.*

Proof. Since R has dimension 3 and $\mathrm{SL}_4(R,I) \cap \mathrm{E}(R,I) = \mathrm{E}_4(R,I)$ the natural map

$$\mathrm{SL}_n(R,I)/\mathrm{E}_n(R,I) \to \mathrm{SK}_1(R,I)$$

is a bijection whenever $n \geq 4$. Now the maximal spectrum of the Excision algebra $\mathbb{Z} \oplus I$ is the union of finitely many subspaces of dimension at most 3 (See [25], 3.19). So we have

$$e_1\mathrm{SL}_{2r+1}(\mathbb{Z} \oplus I) = \mathrm{Um}_{2r+1}(\mathbb{Z} \oplus I) \text{ for all } r \geq 2.$$

Now assume $v \in \mathrm{Um}_{2r}(\mathbb{Z} \oplus I)$, $r \geq 2$ which is stably elementary equivalent to e_1. By elementary operations if necessary we may assume that $v = e_1 \,(\mathrm{mod}\ I)$. Thus $v\alpha = e_1$ for some $\alpha \in \mathrm{SL}_{2r}(\mathbb{Z} \oplus I) \cap \mathrm{E}(\mathbb{Z} \oplus I)$. Going modulo $0 \oplus I$ we have $e_1\overline{\alpha} = e_1$ for $\overline{\alpha} \in \mathrm{E}_{2r}(\mathbb{Z})$. Replacing α by $\alpha\overline{\alpha}^{-1}$ we may assume that $\alpha \in \mathrm{SL}_{2r}(\mathbb{Z} \oplus I, 0 \oplus I) \cap \mathrm{E}(\mathbb{Z} \oplus I, 0 \oplus I)$ (see Lemma 3.3).

Let $\widetilde{v}, \widetilde{\alpha}$ be the image in $\mathrm{Um}_{2r}(R,I), \mathrm{SL}_{2r}(R,I)$ respectively under the maps induced by f (see Definition 2.5). Then $\widetilde{v}\widetilde{\alpha} = e_1$, $\widetilde{\alpha} \in \mathrm{SL}_{2r}(R,I) \cap \mathrm{E}(R,I)$. So $\widetilde{\alpha} \in \mathrm{E}_{2r}(R,I)$ by given hypothesis and

$\widetilde{v}$ is trivial in $\mathrm{MSE}_{2r}(R,I)$. Then by Excision Theorem 2.6, v is also trivial in $\mathrm{MSE}_{2r}(\mathbb{Z}\oplus I)$ i.e. $v\in e_1\mathrm{E}_{2r}(\mathbb{Z}\oplus I)$. Thus we have

$$e_1(SL_{2r}(\mathbb{Z}\oplus I)\cap \mathrm{E}(\mathbb{Z}\oplus I))=e_1\mathrm{E}_{2r}(\mathbb{Z}\oplus I),\ \text{whenever}\, r\geq 2.$$

Now ([17], Theorem 5.2 b, c) shows that $\mathrm{Um}_3(\mathbb{Z}\oplus I)/\mathrm{SL}_3(\mathbb{Z}\oplus I)\cap\mathrm{E}(\mathbb{Z}\oplus I)=W_E(\mathbb{Z}\oplus I)$. We claim that $\mathrm{SL}_3(\mathbb{Z}\oplus I)\cap\mathrm{E}(\mathbb{Z}\oplus I)$ and elementary orbits are same in $\mathrm{Um}_3(\mathbb{Z}\oplus I)$. Choose $u,v\in\mathrm{Um}_3(\mathbb{Z}\oplus I)$ such that $u\alpha=v$ for some $\alpha\in\mathrm{SL}_3(\mathbb{Z}\oplus I)\cap\mathrm{E}(\mathbb{Z}\oplus I)$. By elementary action (*viz.* $\mathrm{E}_3(\mathbb{Z})$ action) if necessary we may assume that $u,v\in\mathrm{Um}_3(\mathbb{Z}\oplus I,0\oplus I)$ and $\alpha\in\mathrm{SL}_3(\mathbb{Z}\oplus I,0\oplus I)\cap\mathrm{E}(\mathbb{Z}\oplus I,0\oplus I)$. Taking images in $\mathrm{Um}_3(R,I)$ as earlier we have $\widetilde{u}\widetilde{\alpha}=\widetilde{v}$ where $\widetilde{\alpha}=\mathrm{SL}_3(R,I)\cap\mathrm{E}(R,I)=\mathrm{SL}_3(R,I)\cap\mathrm{E}_4(R,I)$. Then by Lemma 5.2, $[\widetilde{v}]=[\widetilde{u}]$ in $\mathrm{MSE}_3(R,I)$ and therefore $[v]=[u]$ in $\mathrm{MSE}_3(\mathbb{Z}\oplus I)$ by Excision Theorem 2.6. So we have $\mathrm{MSE}_3(\mathbb{Z}\oplus I)=W_E(\mathbb{Z}\oplus I)$. Hence $\mathrm{MSE}_3(R,I)=\mathrm{MSE}_3(\mathbb{Z}\oplus I)$ has a group structure given by Vaserstein's rule ([17], Theorem 5.2.a). $\square$

Remark 5.4. *Lemma 5.1 and Theorem 5.3 show that* $\mathrm{MSE}_3(A,I)$ *has an abelian Witt group structure (in particular, satisfies* Vv *rule) whenever* A *is a non-singular affine algebra of dimension* 3 *and* I *a principal ideal in* A.

6. A nice group structure on $\mathrm{Um}_d(A)/\mathrm{E}_d(A)$

We first recall the following result in [13].

Theorem 6.1. ([13] Corollary 3.5) *Let* A *be a regular affine algebra of Krull dimension* 3 *over a* C_1 *field* k *which is perfect if its characteristic is* 2 *or* 3. *Then the Vaserstein Symbol* $V:\mathrm{Um}_3(A)/\mathrm{E}_3(A)\longrightarrow W_E(A)$ *is an isomorphism.*

Remark 6.2. *Let* A *be a non-singular affine algebra of dimension* $d, d\geq 3$. *When* $d=3$, *above Theorem together with* ([17], *Theorem 5.2.a*) *says that* $\mathrm{Um}_3(A)/\mathrm{E}_3(A)$ *has a group structure given by* Vv *rule. Theorem 2.9 says so when* $d\geq 4$. *Therefore* $\mathrm{MSE}_d(A)$ *has a group operation* $*$ *defined on it given by* Vv *rule whenever* $d\geq 3$.

Lemma 6.3. ([13] Theorem 5.1)

Let A *be a smooth affine algebra of dimension* $d\geq 3$ *over a perfect* C_1 *field. Let* $1\leq k\leq d$. *Let* $v=(v_1,v_2,\ldots,v_d)\in\mathrm{Um}_d(A)$ *and let* T *be a* $k\times k$ *matrix over* A *with first row* $u=(u_1,u_2,\ldots,u_k)$ *such that* $\overline{det(T)}$ *is a square of a unit in* $A/(v_{k+1},\ldots,v_d)$. *Then*

$$[(v_1,v_2,\ldots,v_k)\cdot T,v_{k+1},\ldots,v_d]=[v]+[u,v_{k+1},\ldots,v_d].$$

In particular, taking $k=d$, *we have* $[v\cdot g]=[v]+[e_1g]$, *for* $g\in SL_d(A)$.

Theorem 6.4. ([4] Lemma 3.3) *Let* S *be a smooth affine surface over an algebraically closed field of characteristic different from* 2, 3. *Then we have* $\mathrm{SL}_2(S)\cap\mathrm{E}(S)=\mathrm{SL}_2(S)\cap\mathrm{E}_3(S)=\mathrm{SL}_2(S)\cap\mathrm{ESp}_4(S)=\mathrm{SL}_2(S)\cap\mathrm{ESp}(S)$.

We shall need the following relative singular version of J. Fasel's observation which can be deduced via ([3], Corollary 4.2 and Corollary 5.3).

Lemma 6.5. ([6]) *Let* A *be an affine threefold over an algebraically closed field. Let* I *be an ideal of* A. *Then* $\mathrm{Um}_4(A,I)=e_1\mathrm{Sp}_4(A,I)$.

Lemma 6.6. *Let* A *be an affine algebra of dimension* 3 *over an algebraically closed field* k *of characteristic different from* 2, 3 *and let* $a\in A$ *be such that* $A/(a)$ *is smooth and* $\dim(A/(a))=2$. *Assume that* $\mathrm{Um}_4(A,(a))=e_1\mathrm{Sp}_4(A,(a))$. *If* $\overline{\sigma}\in\mathrm{SL}_2(A/(a))\cap\mathrm{E}_3(A/(a))$ *then it has a lift* $\sigma\in\mathrm{SL}_2(A)$.

Proof. The argument is similar to that in ([18], Lemma 2.1), (also see ([17], Chapter III)). We recall it for the convenience of the reader.

By Theorem 6.4 $\overline{\sigma} \in \mathrm{SL}_2(A/(a)) \cap \mathrm{E}_3(A/(a)) = \mathrm{SL}_2(A/(a)) \cap \mathrm{ESp}_4(A/(a))$. Therefore $\tau \in \mathrm{ESp}_4(A))$ such that $\overline{\tau} = \overline{\sigma} \perp I_2$. Note that $e_4\tau = e_4 (\mathrm{mod}\ a)$. So by our assumption we have $\delta \in \mathrm{Sp}_4(A, a)$ so that $e_4\tau = e_4\delta$. Let $\varepsilon = \tau\delta^{-1} \in \mathrm{Sp}_4(A)$. Then $e_4\varepsilon = e_4$ and $\overline{\varepsilon} = \overline{\sigma} \perp I_2$. It is easy to see that ε will look like $\begin{pmatrix} \sigma & 0 & * \\ * & 1 & * \\ 0 & 0 & 1 \end{pmatrix}$ for some $\sigma \in \mathrm{SL}_2(A)$. Then σ is a lift of $\overline{\sigma}$. □

Theorem 6.7. *Let A be a smooth affine algebra of dimension 3 over an algebraically closed field k of characteristic not equal to $2, 3$. Then the group structure on the orbit space $\mathrm{MSE}_3(A)$ is nice.*

Proof. A group structure on $\mathrm{MSE}_3(A)$ exists by Theorem 6.1. Let $[v] = [(a, a_1, a_2)]$ and $[w] = [(b, a_1, a_2)]$. We will show that $[v] * [w] = [(ab, a_1, a_2)]$. Applying Swan's version of Bertini's Theorem as in ([21]), we can add a general linear combination of ab, a_1 to a_2 changing it to a_2' and assume that $A/(a_2')$ is a smooth affine surface. Theorem 6.1 shows that $\mathrm{MSE}_3(A)$ has a group structure *viz.*

$$
\begin{aligned}
&[(b, a_1, a_2)] * [(a, a_1, a_2)] \\
&\qquad = [(b, a_1, a_2')] * [(a, a_1, a_2')] \\
&\qquad = [(a(b+p) - 1, (b+p)a_1, a_2')] \qquad \ldots\ldots (1).
\end{aligned}
$$

where p is chosen so that $ap - 1$ belongs to the ideal generated by a_1, a_2'. Let 'overline' denote the image in $\overline{A} := A/(a_2')$ and ms the universal Mennicke symbol. Then, we have

$$
\begin{aligned}
&\mathrm{ms}(\overline{a}(\overline{b} + \overline{p}) - 1, (\overline{b} + \overline{p})\overline{a_1}) \\
&\qquad = \mathrm{ms}(\overline{a}(\overline{b} + \overline{p}) - 1, (\overline{b} + \overline{p}))\mathrm{ms}(\overline{a}(\overline{b} + \overline{p}) - 1, \overline{a_1}) \\
&\qquad = \mathrm{ms}(\overline{a}(\overline{b} + \overline{p}) - 1, \overline{a_1}) \\
&\qquad = \mathrm{ms}(\overline{a}\overline{b}, \overline{a_1}).
\end{aligned}
$$

So there exists $\overline{\sigma} \in \mathrm{SL}_2(\overline{A}) \cap \mathrm{E}_3(\overline{A})$ such that $(\overline{a}(\overline{b} + \overline{p}) - 1, (\overline{b} + \overline{p})\overline{a_1})\overline{\sigma} = (\overline{a}\overline{b}, \overline{a_1})$. By combining Lemma 6.5 and Lemma 6.6, one knows that $\overline{\sigma}$ has a lift $\sigma \in \mathrm{SL}_2(A)$. So we have

$$
\begin{aligned}
&[(ab, a_1, a_2)] \\
&\qquad = [(ab, a_1, a_2')] \\
&\qquad = [(a(b+p) - 1, (b+p)a_1, a_2') \begin{pmatrix} \sigma & 0 \\ 0 & 1 \end{pmatrix}] \\
&\qquad = [(a(b+p) - 1, (b+p)a_1, a_2')] * [e_1 \begin{pmatrix} \sigma & 0 \\ 0 & 1 \end{pmatrix}];\ \text{by Lemma 6.3} \\
&\qquad = [(a(b+p) - 1, (b+p)a_1, a_2')] \qquad \ldots\ldots (2).
\end{aligned}
$$

Hence, the result follows from equations (1) and (2). □

Theorem 6.8. *Let A be a smooth affine algebra over an algebraically closed field k of characteristic not equal to $2, 3$. Then, the group structure on the orbit space $\mathrm{MSE}_d(A)$, $d \geq 3$ is nice, i.e. it is given by the 'coordinate-wise multiplication' formula:*

$$[(a, a_1, a_2, \ldots, a_{d-1})] * [(b, a_1, a_2, \ldots, a_{d-1})] = [(ab, a_1, a_2, \ldots, a_{d-1})].$$

Proof. We shall proceed by induction on d. $d = 3$ is proved in Theorem 6.7. Let $v = (a, a_1, a_2, \ldots, a_{d-1})$, $w = (b, a_1, a_2, \ldots, a_{d-1})$ be such that $v, w \in \mathrm{Um}_d(A)$. Applying R. G. Swan's version of Bertini's Theorem as in ([21]), we can add a general linear combination of $ab, a_1, a_2 \ldots, a_{d-2}$ to a_{d-1} changing it to

a'_{d-1} and assume that $A/(a'_{d-1})$ is a smooth affine algebra of dimension $d-1$. Now we have a group structure on $\mathrm{MSE}_d(A)$ by Remark 6.2. Choose $p \in A$ such that $ap \equiv 1(\mathrm{mod}\ (a_1, a_2, \ldots a'_{d-1}))$. Then

$$\begin{aligned} [w] * [v] &= [(b, a_1, a_2, \ldots, a'_{d-1})] * [(a, a_1, a_2, \ldots, a'_{d-1})] \\ &= [(a(b+p)-1, a_1(b+p), a_2, \ldots, a'_{d-1})] \ldots [1]. \end{aligned}$$

Going modulo a'_{d-1}, we have

$$\begin{aligned} &[(\overline{a}(\overline{b}+\overline{p})-\overline{1}, \overline{a_1}(\overline{b}+\overline{p}), \overline{a_2}, \ldots, \overline{a_{d-2}})] \\ &\qquad = [(\overline{a}, \overline{a_1}, \overline{a_2}, \ldots, \overline{a_{d-2}})] * [(\overline{b}, \overline{a_1}, \overline{a_2}, \ldots, \overline{a_{d-2}})] \quad \text{by Remark 6.2} \\ &\qquad = [(\overline{ab}, \overline{a_1}, \overline{a_2}, \ldots, \overline{a_{d-2}})] \quad \text{by induction hypothesis.} \end{aligned}$$

Therefore,

$$\begin{aligned} [(a(b+p)-1, a_1(b+p), a_2, \ldots, a'_{d-1})] &= [(ab, a_1, a_2, \ldots, a'_{d-1})] \\ &= [(ab, a_1, a_2, \ldots, a_{d-1})] \ldots [2]. \end{aligned}$$

and the result follows from [1] and [2]. $\square$

Corollary 6.9. $\mathrm{MSE}_d(A) = \mathrm{WMS}_d(A) = \mathrm{MS}_d(A)$ *for A satisfying properties as in earlier theorem.*

7. A relative Mennicke–Newman Lemma

We begin by recalling the two cases of the Mennicke–Newman lemma proved by W. van der Kallen (following ([16], Lemma 1.2) and ([1], Lemma 2.4)). We then proceed to prove an analogue of it.

First the relative case:

Lemma 7.1. ([25], *Lemma* 3.4) *Let R be a commutative ring of Krull dimension d. Let v, w be unimodular rows of length $d+1$ relative to an ideal I of R. Then there exist $\varepsilon_1, \varepsilon_2 \in \mathrm{E}_{d+1}(R, I)$ such that $v\varepsilon_1 = (x, a_2, \ldots., a_{d+1})$, $w\varepsilon_2 = (y, a_2, \ldots., a_{d+1})$, with $V(a_2, \ldots, a_{d+1})$ is a union of the closed set $V(I + a_2R + \ldots + a_{d+1}R)$ and finitely many subsets of dimension* 0.

Next the absolute case:

Lemma 7.2. ([27], *Lemma* 3.2) *Let R be a ring of stable dimension $d \le 2n-3$. Let $v, w \in Um_n(R)$. Then there are $\varepsilon_1, \varepsilon_2 \in \mathrm{E}_n(R)$, and $x, y, a_i \in R$, with $x + y = 1$ such that $v\varepsilon_1 = (x, a_2, \ldots, a_n)$, $w\varepsilon_2 = (y, a_2, \ldots, a_n)$.*

The following are relative versions of ([27], Lemma 3.2).

Lemma 7.3. (*Relative Mennicke–Newman*) *Let R be a ring of stable dimension d with $d \le 2n-3$ and I an ideal in R. Let $v, w \in \mathrm{Um}_n(R, I)$. Then there exists $\varepsilon_1, \varepsilon_2 \in \mathrm{E}_n(R, I)$ such that $v\varepsilon_1 = (a_1, a_2, \ldots, a_{n-1}, a)$ and $w\varepsilon_2 = (a_1, a_2, \ldots, a_{n-1}, b)$ such that $a + b$ is a unit modulo $(a_1, a_2, \ldots, a_{n-1})$.*

Proof. Let $v = (a_1, a_2, \ldots, a_n)$, $w = (b_1, b_2, \ldots, b_n) \in \mathrm{Um}_n(R, I)$. Then $(a_1, a_2, a_3 \ldots, a_{n-1}, b_1, b_2, \ldots, b_{n-1}, a_nb_n) \in \mathrm{Um}_{2n-1}(R)$. Since $Sr(R) = 1 + Sd(R) \le 2n-2$, we can find $c_1, c_2, \ldots, c_{n-1}, d_1, d_2, \ldots, d_{n-1} \in R$ such that $(a_1 + c_1a_nb_n, a_2 + c_2a_nb_n, \ldots, a_{n-1} + c_{n-1}a_nb_n, b_1 + d_1a_nb_n, b_2 + d_2a_nb_n, \ldots, b_{n-1} + d_{n-1}a_nb_n) \in \mathrm{Um}_{2n-2}(R, I)$. We add multiples of a_n, b_n *viz.* $c_ia_nb_n$ to a_i and $d_ia_nb_n$ to b_i, $1 \le i \le n-1$ to assume that the the ideals $(a_1, a_2, \ldots, a_{n-1})$ and $(b_1, b_2, \ldots, b_{n-1})$ are comaximal. Now adding a suitable I linear combination of $a_1, a_2, \ldots a_{n-1}$ to a_n and that of $b_1, b_2, \ldots b_{n-1}$ to b_n we can make $b_n - a_n = a_1 - b_1$. Therefore adding last coordinates to the first coordinates we may assume that $a_1 = b_1$. We can do this by $\mathrm{E}_n(A, I)$ action since for any $u = (1 + i_1, i_2, \ldots, i_n) \in \mathrm{Um}_n(R, I)$ we have

$$(1 + i_1 + i_n, i_2, \ldots, i_n) = uE_{n1}(1) = uE_{12}(i_n)E_{n2}(-i_1)E_{21}(1)E_{12}(-i_n)E_{n2}(i_1 + i_2 + i_n)E_{21}(-1).$$

Let $a_1 = b_1 = 1-\lambda, \lambda \in I$. Now by elementary action we change v and w to $v_1 = (a_1, \lambda^2 a_2, \ldots, \lambda^2 a_n)$ and $w_1 = (b_1, \lambda^2 b_2, \ldots, \lambda^2 b_n)$ respectively. Considering the row $(a_1, \lambda^2 a_2, \lambda^2 a_3 \ldots, \lambda^2 a_{n-1}, b_1, \lambda^2 b_2, \lambda^2 b_3, \ldots, \lambda^2 b_{n-1}, \lambda^4 a_n b_n) \in \mathrm{Um}_{2n-1}(R)$ and arguing as in the previous paragraph we change v_1, w_1 to $v_2 = (a'_1, a'_2, \ldots, a'_n)$ and $w_2 = (b'_1, b'_2, \ldots, b'_n)$ respectively by adding multiples of $\lambda^4 a_n b_n$ to the first $n-1$ coordinates such that the ideals $(a'_1, a'_2, \ldots, a'_{n-1})$ and $(b'_1, b'_2, \ldots, b'_{n-1})$ are comaximal. We have $a'_i \equiv b'_i (\mathrm{mod}\ \lambda^2), i = 1, 2, \ldots, n-1$. Now adding suitable I linear combinations of the first $n-1$ coordinates to the last we may assume that $a'_n + b'_n = \lambda$. We note that $a'_i - b'_i = c_i \lambda^2 = c_i \lambda(a'_n + b'_n), c_i \in R, i = 1, 2, \ldots, n-1$. Therefore we can add suitable λ multiples of the last coordinate to the first $n-1$ coordinates to have $a'_i = b'_i$ for $i = 1, 2, \ldots, n-1$. Since $a'_1 \equiv 1(\mathrm{mod}\ \lambda)$, $a'_n + b'_n$ is a unit modulo the ideal generated by the first $n-1$ coordinates. ☐

Lemma 7.4. (*Relative Mennicke–Newman*) *Let R be a commutative noetherian ring and I an ideal in R such that the $max(R)$ is a disjoin union of $V(I)$ and finitely many irreducible closed sets V_i each a noetherian topological space of dimension at most d. Assume $d \leq 2n-3$ and $n \geq 3$. Let $v, w \in \mathrm{Um}_n(R, I)$. Then there exists $\varepsilon_1, \varepsilon_2 \in \mathrm{E}_n(R, I)$ such that $v\varepsilon_1 = (a, a_2, a_3, \ldots, a_n)$ and $w\varepsilon_2 = (b, a_2, a_3, \ldots, a_n)$.*

Proof. Note that if $d = 0, 1$ then any unimodular row of length atleast 3 is elementarily completable. So the result follows obviously. Therefore we shall assume that $2 \leq d \leq 2n-3$. By Remark 2.10 stable dimension of $\mathbb{Z} \oplus I$ is atmost d. We choose $\tilde{v}, \tilde{w} \in \mathrm{Um}_n(\mathbb{Z} \oplus I, 0 \oplus I)$ such that $[\tilde{v}], [\tilde{w}] \in \mathrm{MSE}_n(\mathbb{Z} \oplus I, I)$ correspond to $[v], [w] \in \mathrm{MSE}_n(R, I)$ by F (see Theorem 2.6) respectively. Now by Lemma 7.2 we have $\varepsilon_1, \varepsilon_2 \in \mathrm{E}_n(\mathbb{Z} \oplus I)$ such that $\tilde{v}\varepsilon_1 = (\tilde{a}, \tilde{a_2}, \ldots, \tilde{a_n})$ and $\tilde{w}\varepsilon_2 = (\tilde{b}, \tilde{a_2}, \ldots, \tilde{a_n}), \tilde{a} + \tilde{b} = 1$. Assume $\tilde{a_i} \equiv n_i$ (mod I) for $i \geq 2$ and $c = g.c.d(n_i)$. Then by further elementary action (infact by $\mathrm{E}_n(\mathbb{Z})$ action) we may assume that $(\tilde{a_2}, \tilde{a_3}, \ldots, \tilde{a_n}) \equiv (c, 0, \ldots, 0)(\mathrm{mod}\ I)$. Adding $\tilde{a}\tilde{b}$ to the third coordinate we have $(\tilde{a_2}, \tilde{a_3}, \ldots, \tilde{a_n}) \equiv (c, e, \ldots, 0)(\mathrm{mod}\ I)$ such that $g.c.d(c, e) = 1$. Now we shall add suitable $\mathbb{Z}$ linear combination of $\tilde{a_2}, \tilde{a_3}$ to $\tilde{a_1}$ to have $\tilde{a} \equiv \tilde{b} \equiv 1(\mathrm{mod}\ I)$. Then we shall add a suitable $\mathbb{Z}$ multiples $\tilde{a}\tilde{b}$ to the rest and have $(\tilde{a}, \tilde{a_2}, \ldots, \tilde{a_n}) \equiv (\tilde{b}, \tilde{a_2}, \ldots, \tilde{a_n}) \equiv e_1(\mathrm{mod}\ I)$.

Thus we have $\varepsilon'_1, \varepsilon'_2 \in \mathrm{E}_n(\mathbb{Z} \oplus I)$ such that $\tilde{v}\varepsilon'_1 = \tilde{v}' = (\tilde{a}, \tilde{a_2}, \ldots, \tilde{a_n}), \tilde{w}\varepsilon'_2 = \tilde{w}' = (\tilde{b}, \tilde{a_2}, \ldots, \tilde{a_n})$. Note that $\tilde{v}, \tilde{v}', \tilde{w}, \tilde{w}' \in \mathrm{Um}_n(\mathbb{Z} \oplus I, 0 \oplus I)$. So by Excision Theorem 2.6 we have $\varepsilon''_1, \varepsilon''_2 \in \mathrm{E}_n(\mathbb{Z} \oplus I, 0 \oplus I)$ such that $\tilde{v}\varepsilon''_1 = \tilde{v}', \tilde{w}\varepsilon''_2 = \tilde{w}'$. Now the result follows taking projection onto R under $f : \mathbb{Z} \oplus I \to R$ defined by $f(n, i) = n + i$. ☐

8. Niceness of relative orbit space group

In this section we shall first establish the relative analogue of results in the previous section. By Remark 2.10 and 5.4 we know that $\mathrm{MSE}_d(A, I), d \geq 3$ has a group structure given by Vv rule whenever A is a nonsingular affine algebra.

Theorem 8.1. *Let A be a smooth affine algebra of dimension 3 over an algebraically closed field k of characteristic not equal to 2, 3 and I a principal ideal in A. Then the group structure on the orbit space $\mathrm{Um}_3(A, I)/\mathrm{E}_3(A, I)$ is nice.*

Proof. By Remark 5.4 we already have a group structure on $\mathrm{MSE}_3(A, I)$ given by Vv rule . Let $[v] = [(a_1, a_2, a)]$, $[w] = [(a_1, a_2, b)]$, $a_1 = 1-\lambda, \lambda \in I$. We shall show that $[w] * [v] = [(a_1, a_2, ab)]$. Note that $(a_1, \lambda a_2, \lambda ab) \in \mathrm{Um}_3(A, I)$ Applying Swan's version of Bertini's Theorem in ([21]), we can add a general linear combination of $\lambda ab, \lambda a_2$ to a_1 changing it to a'_1 and assume that $A/(a'_1)$ is a smooth affine surface. Let $a'_1 = 1 - \eta, \eta \in I$. Now by Vv rule we have

$$[w] * [v] = [(a'_1, a_2, b)] * [(a'_1, a_2, a)] = [(a'_1, a_2(b + p), a(b + p) - \eta)].$$

where $(b_1, b_2, p) \in \mathrm{Um}_3(A, I)$ such that $a_1 b_1 + a_2 b_2 + ap = 1$. Now in $\overline{A} := A/(a'_1)$ we have $\overline{\eta} = 1$. Arguing as in Theorem 6.7 we have $\overline{\sigma} \in \mathrm{SL}_2(\overline{A}) \cap \mathrm{E}_3(\overline{A})$ such that $(\overline{a_2}(\overline{b} + \overline{p}), \overline{a}(\overline{b} + \overline{p}) - \overline{\eta})\overline{\sigma} = (\overline{a_2}, \overline{a}\overline{b})$.

By Lemma 6.6, $\overline{\sigma}$ has a lift $\sigma \in \mathrm{SL}_2(A)$. Clearly $(a_2(b+p), a(b+p)-\eta)\sigma \equiv (a_2, ab)(\text{mod } Ia_1')$. So

$$\begin{aligned}
[(a_1', a_2, ab)] &= [(a_1', a_2(b+p), a(b+p)-\eta)\begin{pmatrix}1 & 0\\ 0 & \sigma\end{pmatrix}] \\
&= [(a_1', a_2(b+p), a(b+p)-\eta)\begin{pmatrix}1 & 0\\ 0 & \eta\sigma\end{pmatrix}] \\
&= [(a_1', a_2(b+p), a(b+p)-\eta)] * [(a_1', e_1\eta\sigma)];\ \text{by } Vv \text{ rule} \\
&= [(a_1', a_2(b+p), a(b+p)-\eta)].
\end{aligned}$$

The last equality is true since $(a_1', e_1\eta\sigma) = e_1(1 \perp \sigma)^{-1}E_{21}(1)E_{12}(\eta)E_{21}(-1)(1 \perp \sigma) \in e_1\mathrm{E}_3(A, I)$. Therefore we have

$$\begin{aligned}
[(a_1, a_2, b)] * [(a_1, a_2, a)] &= [(a_1', a_2, b)] * [(a_1', a_2, a)] \\
&= [(a_1', a_2(b+p), a(b+p)-\eta)] \\
&= [(a_1', a_2, ab)] \\
&= [(a_1, a_2, ab)].
\end{aligned}$$

□

Theorem 8.2. *Let A be a smooth affine algebra over an algebraically closed field k of characteristic not equal to $2, 3$ of dimension $d \geq 4$ and I an ideal in A. Then, the group structure on the orbit space $\mathrm{MSE}_d(A, I)$ is nice, i.e. it is given by the 'coordinate-wise multiplication' formula:*

$$[(a_1, a_2, \ldots, a_{d-1}, a)] * [(a_1, a_2, \ldots, a_{d-1}, b)] = [(a_1, a_2, \ldots, a_{d-1}, ab)].$$

Proof. Let $v = (a_1, a_2, \ldots, a_{d-1}, a)$, $w = (a_1, a_2, \ldots, a_{d-1}, b)$ be such that $v, w \in \mathrm{Um}_d(A, I)$, $a_1 = 1-\lambda$. Then $(a_1, \lambda a_2, \ldots, \lambda a_{d-1}, \lambda ab) \in \mathrm{Um}_n(A, I)$. Applying R. G. Swan's version of Bertini's Theorem in ([21]) we can add a general linear combination of $\lambda ab, \lambda a_{d-1}, \lambda a_{d-2} \ldots, \lambda a_2$ to a_1 changing it to a_1' and assume that $A/(a_1')$ is a smooth affine algebra of dimension $d-1$. Let $a_1' = 1-\eta$. Note that in $\overline{A} = A/a_1'$ we have $\overline{\eta} = 1$. We already have a group structure on $\mathrm{MSE}_d(A, I), d \geq 4$ by Remark 2.10 . We choose $(b_1, b_2, \ldots, b_{d-1}, p) \in \mathrm{Um}_d(A, I)$ such that $a_1'b_1 + \Sigma_{i=2}^{d-1} a_ib_i + ap = 1$. Then

$$\begin{aligned}
[w] * [v] &= [(a_1', a_2, \ldots, a_{d-1}, b)] * [(a_1', a_2, \ldots, a_{d-1}, a)] \\
&= [(a_1', a_2, \ldots, a_{d-2}, a_{d-1}(b+p), a(b+p)-\eta))].
\end{aligned}$$

Going modulo a_1', we have

$$\begin{aligned}
[(\overline{a_2}, \ldots, \overline{a_{d-2}}, \overline{a_{d-1}}(\overline{b}+\overline{p}), \overline{a}(\overline{b}+\overline{p})-\overline{\eta}))] &= [(\overline{a_2}, \ldots, \overline{a_{d-1}}, \overline{b})] * [(\overline{a_2}, \ldots, \overline{a_{d-1}}, \overline{a})] \quad \text{by Remark 6.2} \\
&= [(\overline{a_2}, \overline{a_3}, \ldots, \overline{a_{d-1}}, \overline{ab},)] \quad \text{by Theorems 6.7, 6.8.}
\end{aligned}$$

Therefore we have $(a_2, \ldots, a_{d-2}, a_{d-1}(b+p), a(b+p)-\eta))\alpha \equiv (a_2, \ldots, a_{d-1}, ab)(\text{mod } Ia_1')$ for some $\alpha \in \mathrm{E}_{d-1}(I)$. So

$$\begin{aligned}
[(a_1', a_2, \ldots, a_{d-2}, a_{d-1}(b+p), a(b+p)-\eta))] &= [(a_1', a_2, \ldots, a_{d-1}, ab)] \\
&= [(a_1, a_2, \ldots, a_{d-1}, ab)];\ \text{in } \mathrm{MSE}_d(A, I).
\end{aligned}$$

□

As a consequence one can deduce as in Corollary 4.6:

Corollary 8.3. *Let A be a smooth affine algebra of dimension $d \geq 3$ over an algebraically closed field k, with* char $k \neq 2, 3$. *I satisfies conditions given in Theorems 8.1 and 8.2 depending on $d = 3$ or $d > 3$ respectively. Let $\sigma \in \mathrm{SL}_d(A, I) \cap \mathrm{E}_{d+1}(A, I)$. Then, $[e_1\sigma] = [e_1]$ in $\mathrm{MSE}_d(A, I)$ i.e. $e_1\sigma$ is relatively elementarily completable to e_1.*

Corollary 8.4. $\mathrm{MSE}_d(A, I) = \mathrm{WMS}_d(A, I) = \mathrm{MS}_d(A, I)$ *where A and I satisfy conditions given in Theorems 8.1 and 8.2 depending on $d = 3$ or $d > 3$ respectively.*

Question 8.5. *Let R be a local ring of Krull dimension $d \geq 2$. Let I be an ideal of R. Is the group structure on $\mathrm{MSE}_d(R[X], I[X])$ nice?*

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